BIBLIOGRAPHY

1. Lifshits I. M. and Rozentsveig L. N., Theory of elastic properties of polycrystals. ZhETF Vol. 16, No. 11, 1946.
2. Krŏner E., Elastic moduli of perfectly disordered composite materials. J. Mech. Phys. Solids Vol. 15, No. 5, 1967.
3. Bolotin V. V. and Moskalenko V. N., The problem of determination of elastic constants of a microheterogeneous medium. PMTF, No. 1, 1968.
4. Bolotin V. V. and Moskalenko V. N., Computation of macroscopic constants of strongly isotropic composite materials. Izv. Akad. Nauk SSSR, MTT, No. 3, 1969.
5. Fokin A. G. and Shermergor T. D., Computation of effective elastic moduli for composite materials with consideration of multiparticle interactions. PMTF No. 1, 1969.
6. Novozhilov V. V., The relation between mathematical expectations of stress and strain tensors in statistically isotropic homogeneous elastic bodies. PMM Vol. 34, No. 1, 1970.
7. Green A. E. and Rivlin R. S., Multipolar continuum mechanics. Arch. Rat. Mech. Anal. Vol. 17, No. 2, 1964.
8. Krŏner E., Elasticity theory of materials with long-range cohesive forces. Pergamon Press, Internat. J. Solids and Structures, Vol. 3, No. 5, 1967.
9. Fokin A. G. and Shermergor T. D., Correlation functions of an elastic field of quasi-isotropic solid bodies. PMM Vol. 32, No. 4, 1968.

Translated by B. D.

REPRESENTATION OF THE RESOLVENTS OF OPERATORS OF VISCOELASTICITY IN TERMS OF SPECTRAL DISTRIBUTION FUNCTIONS

PMM Vol. 35, No. 4, 1971, pp. 750-759

V. G. GROMOV
(Rostov-on-Don)

(Received June 30, 1970)

A general representation of the resolvent in terms of a reduced distribution function of a viscoelastic spectrum of the initial kernel is obtained. The method is illustrated using the widely known operators of viscoelasticity. Resolvents are constructed for the generalized fractional exponential kernel and the logarithmic kernels.

Viscoelastic behavior of the real bodies (polymers in particular) can also be described [1 - 3] by other equations containing the Volterra operator

$$P^*(\cdots) = \int_0 P(t-\tau)(\cdots)\,d\tau \tag{0.1}$$

where $P(t)$ is the operator kernel. If $I - \varkappa P^*, -\infty < \varkappa < \infty$ is a complete operator of viscoelasticity, then the operator

$$(I - \varkappa P^*)^{-1} = I + \varkappa R_\varkappa^* \tag{0.2}$$

is the inverse operator and $R_\varkappa^*$ is the resolvent operator for P^*, and the kernel $R_\varkappa(t)$ is the resolvent for $P(t)$. General problems concerned with the invertibility of (0.2) are well worked out and the only question arising is that of the actual construction of the inverse operator.

The feasibility of explicit construction of a resolvent from the given kernel is of essential importance in the theory of viscoelasticity [3] and until recently [4-5] this was borne in mind when constructing the algebra of the Volterra operators (in connection with solutions of the boundary value problems). Although this was later [6-7] shown not to be necessary, nevertheless the problem of constructing a resolvent continues to be of interest.

Two classical methods of constructing the resolvent for the operator (0.1) exist. The symbolic method of Volterra, and the method of operational calculus. The first method is based on formal expansion of (0.2) into a series in the powers of the parameter $\varkappa$. We then have

$$R_\varkappa^* = \sum_{n=1}^{\infty} \varkappa^{n-1} P^{*n} \tag{0.3}$$

The above expansion is called the Neumann series for the operator $R_\varkappa^*$. Here the operator P^* raised to the n-th power denotes a new operator whose kernel has been iterated $(n-1)$ times [8]

$$P^{*n}(\cdots) = \int_0^t P_n(t-\tau)(\cdots)\,d\tau, \qquad P_n(t-\tau) = \int_\tau^t P_{n-1}(t-s)\,P(s-\tau)\,ds \tag{0.4}$$

Two kinds of difficulties arise in practice. Solution of the iterated integrals, and construction of the general term of (0.3). With few exceptions these difficulties cannot be overcome. The method of operational calculus is based on integral transformation of the operators P^* and $R_\varkappa^*$. Since in the present case the latter are convolution operators [9], we can denote their functional Laplace transforms by $P^\circ(p)$ and $R_\varkappa^\circ(p)$ and transform the operator identity (0.2) into the functional identity

$$[1 - \varkappa P^\circ(p)]\,[1 + \varkappa R_\varkappa^\circ(p)] = 1 \tag{0.5}$$

from which we obtain

$$R_\varkappa^\circ(p) = \frac{P^\circ(p)}{1 - \varkappa P^\circ(p)} = \sum_{n=1}^{\infty} \varkappa^{n-1} P^{\circ n}(p) \tag{0.6}$$

The problem of constructing the resolvent is now reduced to that of inverting (0.6). However, even the well developed methods [9] often fail to cope with the problem of inversion. Moreover, cases exist when not only the problem of inversion, but also the problem of constructing the functional transform $P^\circ(p)$ of the operator in its explicit form is met with the difficulties mentioned above.

1. The distribution functions method. The class of the viscoelastic operators is characterized by the fact that each operator can be put in one-to-one correspondence with a nonnegative spectrum (continuous or discrete) possessing a positive distribution function.

Depending on the meaning assigned to the operator, the parts of these spectra can be played by the relaxation spectrum, the delay spectrum [1 - 3] and others. Assuming e.g. that $\varkappa > 0$ in (0.2) we find that P^* now becomes the relaxation operator and $R_\varkappa^*$. is the delay (creep) operator. The corresponding complete operators acting on the unit function will now become the relaxation function $\varphi(t)$ and the creep function $\psi_\varkappa(t)$. The latter functions can also be written in terms of the viscoelastic spectra [1 - 3]

$$\varphi(t) = \int_0^\infty H(\gamma) e^{-t/\gamma} d\gamma, \qquad \psi_\varkappa(t) = \int_0^\infty Q_\varkappa(\gamma)(1 - e^{-t/\gamma}) d\gamma \tag{1.1}$$

Here $H(\gamma)$ and $Q_\varkappa(\gamma)$ are the distribution functions of the relaxation time spectra and the creep time spectra. Since these functions cannot be determined experimentally, they are chiefly used to aid the theoretical interpretations of the properties of viscoelasticity [10 - 12]. All the same, they may be utilized in the problem of constructing the resolvents.

By definition, the relaxation and creep functions are

$$P(t) = -\frac{1}{\varkappa}\frac{d\varphi}{dt}, \qquad R_\varkappa(t) = \frac{1}{\varkappa}\frac{d\psi_\varkappa}{dt} \tag{1.2}$$

Using the representation (1.1) we can obtain from (1.2) the following spectral representations for the kernel and the resolvent:

$$P(t) = \frac{1}{\varkappa}\int_0^\infty h(\lambda) e^{-\lambda t} d\lambda, \qquad R_\varkappa(t) = \frac{1}{\varkappa}\int_0^\infty q_\varkappa(\lambda) e^{-\lambda t} d\lambda \tag{1.3}$$

The reduced distribution functions given here

$$h(\lambda) = \lambda^{-1} H(\lambda^{-1}), \qquad q_\varkappa(\lambda) = \lambda^{-1} Q(\lambda^{-1}), \qquad \lambda = \gamma^{-1} \tag{1.4}$$

are constructed from the original ones by applying the inversion transformation.

We note that (1.3) represents the particular case of the Bochner representation for the positive-definite functions [13]. The idea behind the distribution function method is that the problem of constructing the resolvent $R_\varkappa(t)$ from the given kernel $P(t)$ can be reduced with the help of (1.3) to the problem of constructing the reduced distribution function $q_\varkappa(\lambda)$.

Let us consider the functional transforms $P^\circ(p)$ and $R_\varkappa^\circ(p)$ of the operators P^* and $R_\varkappa^*$ Using (1.3) we can easily show that

$$P^\circ(p) = \frac{1}{\varkappa}\int_0^\infty \frac{h(\lambda)}{p+\lambda} d\lambda, \qquad R_\varkappa^\circ(p) = \frac{1}{\varkappa}\int_0^\infty \frac{q_\varkappa(\lambda)}{p+\lambda} d\lambda \tag{1.5}$$

By virtue of the integrability of $h(\lambda)$ and $q(\lambda)$ the functions $P^\circ(p)$ and $R_\varkappa^\circ(p)$ are well-defined and continuous over the whole plane of the complex variable $p = \alpha + i\beta$, except at the points $p = -\alpha$, $\alpha > 0$ lying on the negative part of the real axis. Let us find the limit value of the functions at these points. For simplicity we shall consider the case when the point is approached along the direction parallel to the imaginary axis, e.g.

$$\lim_{p \to -\alpha} P^\circ(p) = \lim_{\beta \to 0} P^\circ(\alpha \pm i\beta) \tag{1.6}$$

The upper sign corresponds to the case when the real axis is approached from the upper semiplane, and the lower sign to the approach from the lower semiplane. We further

find

$$\varkappa P^{\circ}(-\alpha \pm i\beta) = \int_0^{\infty} \frac{\lambda-\alpha}{(\lambda-\alpha)^2+\beta^2} h(\lambda)\, d\lambda \mp i \int_0^{\infty} \frac{\beta}{(\lambda-\alpha)^2+\beta^2} h(\lambda)\, d\lambda \qquad (1.7)$$

We know that

$$\lim_{\beta\to 0} \frac{\beta}{(\lambda-\alpha)^2+\beta^2} = \pi\delta(\lambda-\alpha), \quad \delta(x) \text{ is the delta function.} \qquad (1.8)$$

Therefore, as $\beta \to 0$ we obtain from (1.7)

$$\varkappa P_{\pm}^{\circ}(-\alpha) = \int_0^{\infty} \frac{h(\lambda)}{\lambda-\alpha} d\lambda \mp i\pi h(\alpha) \qquad (1.9)$$

Here $P_{+}^{\circ}(-\alpha)$ and $P_{-}^{\circ}(-\alpha)$ denote the respective limits from the upper and the lower semiplane. From (1.9) it follows that the function $P^{\circ}(p)$ undergoes a jump

$$\Delta P^{\circ}(-\alpha) = P_{+}^{\circ}(-\alpha) - P_{-}^{\circ}(-\alpha) = -2\pi i \varkappa^{-1} h(\alpha) \qquad (1.10)$$

during the passage across the negative part of the real axis.

The magnitude of this jump is determined (with the accuracy of up to the multiplier term) by the distribution function $h(\alpha)$. This makes possible the use of the elementary methods to obtain $h(\alpha)$ from the given kernel $l'(t)$

$$h(\alpha) = -\frac{\varkappa}{2\pi i} \Delta P_0(-\alpha) \qquad (1.11)$$

The relation (1.11) can be simplified. Assuming

$$P_{\pm}^{\circ}(-\alpha) = \operatorname{Re} P_{\pm}^{\circ}(-\alpha) + i \operatorname{Im} P_{\pm}^{\circ}(-\alpha) \qquad (1.12)$$

we obtain from (1.9)

$$\operatorname{Re} P_{+}^{\circ}(-\alpha) = \operatorname{Re} P_{-}^{\circ}(-\alpha), \qquad \operatorname{Im} P_{+}^{\circ}(-\alpha) = -\operatorname{Im} P_{-}^{\circ}(-\alpha) \qquad (1.13)$$

Then

$$\Delta P^{\circ}(-\alpha) = \pm 2i \operatorname{Im} P_{\pm}^{\circ}(-\alpha) \qquad (1.14)$$

and

$$h(\alpha) = \mp \frac{\varkappa}{\pi} \operatorname{Im} P_{\pm}^{\circ}(-\alpha) \qquad (1.15)$$

The order of the upper and lower signs must be observed here. In exactly the same manner we find, that

$$\varkappa R_{\varkappa\pm}^{\circ}(-\alpha) = \int_0^{\infty} \frac{q_{\varkappa}(\lambda)}{\lambda-\alpha} d\lambda \mp i\pi q_{\varkappa}(\alpha), \quad q_{\varkappa}(\alpha) = -\frac{\varkappa}{\pi} \operatorname{Im} R_{\varkappa-}^{\circ}(-\alpha) \qquad (1.16)$$

The improper integrals in (1.9) and (1.16) are used here in the sense of their principal values.

The kernel $P(t)$ and the resolvent $R_{\varkappa}(t)$ are connected by a functional identity [8] which follows from the operator identity (0.2). This implies that the reduced distribution functions $h(\alpha)$ and $q_{\varkappa}(\alpha)$ should also be connected by some definite relation. This relation can be obtained most simply from (0.5) after completing the limiting passage to the negative part of the real axis. Using the limiting values $P_{\pm}^{\circ}(-\alpha)$ and $R_{\varkappa\pm}^{\circ}(-\alpha)$ we obtain

$$J_1 J_2 + J_1 - J_2 + \pi^2 h q_{\varkappa} = 0, \quad -J_1 q_{\varkappa} - J_2 h + q_{\varkappa} - h = 0 \qquad (1.17)$$

Here J_1 and J_2 denote the improper integrals in (1.9) and (1.16) respectively. These equations make it possible to express one distribution function in the terms of another distribution function. For example, assume that $q_\varkappa$ and J_2 are unknown. Then they can be determined from the system

$$(1-J_1)J_2+\pi^2hq_\varkappa=J_1, \qquad -hJ_2+(1-J_1)q_\varkappa=h \tag{1.18}$$

from which we have

$$q_\varkappa(\alpha)=h(\alpha)\left\{\left[1-\int_0^\infty\frac{h(\lambda)}{\lambda-\alpha}\,d\lambda\right]^2+\pi^2h^2(\alpha)\right\}^{-1} \tag{1.19}$$

Writing (1.17) from h and J_1, we similarly obtain

$$h(\alpha)=q_\varkappa(\alpha)\left\{\left[1+\int_0^\infty\frac{q_\varkappa(\lambda)}{\lambda-\alpha}\,d\lambda\right]^2+\pi^2q_\varkappa^2(\alpha)\right\}^{-1} \tag{1.20}$$

Relations (1.19) and (1.20) make it possible to use a known distribution function to obtain another distribution function, and they correspond exactly to the classical relations used for recalculating the relaxation and delay spectra [1, 10].

We now have everything necessary for constructing a scheme of obtaining the resolvent. We use the given kernel $P(t)$ to obtain the reduced distribution function $h(\alpha)$. Depending on the conditions used in defining $P(t)$, we can either use the relations (1.15), or (1.1) - (1.3). When (1.15) is used, the imaginary part of $P^\circ(p)$ is computed on the negative part of the real axis. The choice of sign is governed by the choice of the argument (π or $-\pi$) and $P^\circ(p)$ is understood to be the analytic continuation of the ordinary Laplace transform of the kernel $P(t)$ over the whole complex p-plane. The distribution function $q_\varkappa(\alpha)$ is found from (1.19). Finally, (1.3) is used to obtain an expression for the resolvent $R_\varkappa(t)$

All these operations are relatively simple and, what is more important, they are given in the explicit form. This simplifies both the analytic construction of the resolvent and its numerical analysis.

2. Application of the method. We shall illustrate the above method by applying it to the widely known viscoelastic operators. At the same time we shall construct certain resolvents, which have not up to now been obtained. We begin with the simplest case.

1. Exponential operator. Let

$$P^*(\cdots)=\Im^*(\cdots)=\int_0^t e^{-\mu(t-\tau)}(\cdots)\,d\tau \qquad \text{or} \qquad P(t)=e^{-\mu t}, \quad \mu>0 \tag{2.1}$$

We find that

$$\Im^\circ(p)=\frac{1}{p+\mu}, \qquad \operatorname{Im}\frac{1}{-\alpha+\mu}=0 \tag{2.2}$$

The latter means that the operator spectrum is discrete. From (1.3) follows

$$e^{-\mu t}=\frac{1}{\varkappa}\int_0^\infty h(\lambda)\,e^{-\lambda t}\,d\lambda, \qquad h(\lambda)=\varkappa\delta(\lambda-\mu) \tag{2.4}$$

Here $\lambda=\mu$ is a point of the spectrum. Further,

$$q_\varkappa(\alpha)=\frac{\varkappa\delta(\alpha-\mu)}{[1-\varkappa/\mu-\alpha]^2+\pi^2\varkappa^2\delta^2(\alpha-\mu)}=\varkappa\delta[\alpha-(\mu-\varkappa)] \tag{2.5}$$

since (2.5), similarly to (1.8), can be represented by the following limit [1]:

$$q_\varkappa(\alpha) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\frac{\varepsilon}{M^2(\alpha)+\varepsilon^2} = \frac{\delta(\alpha-\alpha_*)}{|M'(\alpha_*)|}, \qquad M(\alpha_*)=0 \tag{2.6}$$

$$M(\alpha) = 1 - \frac{\varkappa}{\mu-\alpha}, \qquad |M'(\alpha_*)| = \frac{1}{\varkappa}, \qquad \alpha_* = \mu - \varkappa \tag{2.7}$$

The resolvent has the form

$$R_\varkappa(t) = \int_0^\infty \delta[\lambda - (\mu-\varkappa)]\, e^{-\lambda t}\, d\lambda = e^{-(\mu-\varkappa)t} \tag{2.8}$$

It can easily be shown that the same result is obtained by direct inversion of the operator $\Im^*(\cdots)$.

2. Generalized fractional exponential operator. Consider an operator of the form

$$P^*(\cdots) = E^*(\cdots) = \frac{1}{\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1} e^{-\mu(t-\tau)}(\cdots)\, d\tau \qquad (0<\nu\leqslant 1) \tag{2.9}$$

We find

$$E^\circ(p) = \frac{1}{(p+\mu)^\nu}, \qquad \operatorname{Im} E_+^\circ(-\alpha) = \begin{cases} 0, & \alpha<\mu \\ -\dfrac{\sin\pi\nu}{(\alpha-\mu)^\nu}, & \alpha>\mu \end{cases} \tag{2.10}$$

which, together with (1.15), yield

$$h(\alpha) = 0 \quad (\alpha<\mu), \qquad h(\alpha) = |\varkappa\frac{\sin\pi\nu}{\pi}\frac{1}{(\alpha-\mu)^\nu} \quad (\alpha>\mu) \tag{2.11}$$

i.e. additional multiplication of the Abel kernel by an exponential term corresponds to a direct displacement of the spectrum. We obtain $q_\varkappa(\alpha)$ with the help of the following known integral [14]

$$\int_0^\infty \frac{x^\nu}{x+a}dx = \begin{cases} \pi a^{-\nu}\csc(1-\nu)\pi & (a>0) \\ -\pi a^{-\nu}\operatorname{ctg}(1-\nu)\pi & (a<0) \end{cases} \tag{2.12}$$

Then

$$\int_0^\infty \frac{h(\lambda)}{\lambda-\alpha}d\lambda = \frac{\varkappa\sin\pi\nu}{\pi}\int_\mu^\infty \frac{(\lambda-\mu)^{-\nu}}{\lambda-\alpha}d\lambda = \frac{\varkappa\sin\pi\nu}{\pi}\int_0^\infty \frac{\gamma^{-\nu}d\gamma}{\gamma+\mu-\alpha} = \frac{\varkappa\cos\pi\nu}{(\alpha-\mu)^\nu} \tag{2.13}$$

Inserting (2.11) and (2.13) into (1.16) we obtain

$$q_\varkappa(\alpha) = \begin{cases} 0, & (\alpha<\mu) \\ \pi^{-1}\varkappa(\alpha-\mu)^\nu\sin\pi\nu\,[(\alpha-\mu)^{2\nu} - 2\varkappa(\alpha-\mu)^\nu\cos\pi\nu + \varkappa^2]^{-1}, & (\alpha>\mu) \end{cases} \tag{2.14}$$

The following expression for the resolvent follows from (1.3)

$$R_\varkappa(t) = \frac{\sin\pi\nu}{\pi}\int_\mu^\infty \frac{(\lambda-\mu)^\nu e^{-\lambda t}}{(\lambda-\mu)^{2\nu} - 2\varkappa(\lambda-\mu)^\nu\cos\pi\nu + \varkappa^2}\, d\lambda \tag{2.15}$$

Neumann type expansion is obtained when the integrand expression is written in the powers of $\varkappa$. Denoting $\lambda - \mu = \rho$ we find

$$q_\varkappa(\rho) = \frac{1}{2\pi i}\left(\frac{\rho_1}{\rho-\rho_1} - \frac{\rho_2}{\rho^\nu-\rho_2}\right) \tag{2.16}$$

Here $\rho_1 = \varkappa e^{i\pi\nu}$ and $\rho_2 = \varkappa e^{-i\pi\nu}$ are the roots of the denominator. Taking this into account in (2.16), we obtain

$$q_\varkappa(\rho) = \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\rho^\nu}{\varkappa}\right)^n \sin n\pi\nu, \quad \rho < |\varkappa|^{1/\nu}$$

$$q_\varkappa(\rho) = \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\varkappa}{\rho^\nu}\right)^n \sin n\pi\nu, \quad \rho > |\varkappa|^{1/\nu} \tag{2.17}$$

and the integral

$$\int_0^\infty q_\varkappa(\rho)\, e^{-\rho t}\, d\rho = \frac{1}{\pi}\sum_{n=1}^{\infty}\varkappa^n t^{n\nu-1}\Gamma(1-n\nu)\sin n\pi\nu \tag{2.18}$$

Hence from (2.15) we obtain

$$R_\varkappa(t) = e^{-\mu t}\vartheta_\nu(\varkappa, t) \tag{2.19}$$

where ϑ_ν is a fractional-power exponential function [4, 5].

3. Bronskii-Slonimskii operator. The operator

$$P^\rho(\cdots) = \frac{1}{\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1} e^{-\mu(t-\tau)^\gamma}(\cdots)\, d\tau, \quad \gamma > 0 \tag{2.20}$$

represents a further generalization of the exponential operator. The power term appearing in the exponential index seriously complicates the computations, and no unique analytic expression for the functional transform $P^\rho(p)$ exists on the whole of the p-plane. Indeed, writing the exponential function in the form of a series we find

$$P^\rho(p) = \frac{1}{p^\nu}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n\gamma+\nu)}{\Gamma(\nu)\,n!}\left(\frac{\mu}{p^\gamma}\right)^n \tag{2.21}$$

and the series converges for $|p| > \mu^{1/\gamma}$, i.e. outside the circle of radius $\mu^{1/\gamma}$. The analytic continuation of $P^\rho(p)$ into the interior of the circle is constructed proceeding from the following integral representation (by definition)

$$P^\rho(p) = \frac{1}{\Gamma(\nu)}\int_0^\infty t^{\nu-1} e^{-\mu t^\gamma} e^{-pt}\, dt \tag{2.22}$$

Expanding (2.22) into a series in the powers of p about the origin, we find

$$P^\rho(p) = \frac{1}{\gamma\mu^{1/\gamma}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\nu)\,n!}\,\Gamma\left(\frac{\nu+n}{\gamma}\right)\left(\frac{p}{\mu^{1/\gamma}}\right)^n \tag{2.23}$$

with the region of convergence $|p| < \mu^{1/\gamma}$. Thus the relations (2.21) and (2.22) give the analytic expression for $P^\nu(p)$ at all points of the p-plane except for those on the interval $(-\infty, -\mu^{1/\gamma}]$ lying on the negative part of the real axis. From this in turn we find

$$\operatorname{Im} P_+^\rho(-\alpha) = 0, \qquad 0 < \alpha < \mu^{1/\gamma}$$

$$\operatorname{Im} P_+^\rho(-\alpha) = -\frac{1}{\alpha^\nu}\sum_{n=0}^{\infty}\frac{\Gamma(n\gamma+\nu)\sin(n\gamma+\nu)\pi}{\Gamma(\nu)\,n!}\left(\frac{\mu}{\alpha^\gamma}\right)^n, \quad \alpha > \mu^{1/\gamma} \tag{2.24}$$

Now according to (1.15) we have

$$h(\alpha) = 0, \qquad 0 < \alpha < \mu^{1/\gamma}$$

$$h(\alpha) = \frac{\varkappa}{\pi\alpha^\nu}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\theta_n)\sin\theta_n\pi}{\Gamma(\nu)\,n!}\left(\frac{\mu}{\alpha^\gamma}\right)^n, \quad \alpha > \mu^{1/\gamma}, \quad \theta_n = \gamma n + \nu \tag{2.25}$$

Further

$$J(\alpha)=\int_0^\infty \frac{h(\lambda)}{\lambda-\alpha}\,d\lambda=\frac{\varkappa}{\pi}\sum_{n=0}^{\infty}(-\mu)^n\frac{\Gamma(\theta_n)\sin\theta_n\pi}{\Gamma(\nu)\,n!}\,J_n \tag{2.26}$$

$$J_n=\int_{\mu^{1/\gamma}}^{\infty}\frac{\lambda^{-\theta_n}}{\lambda-\alpha}\,d\lambda=\int_0^\infty\frac{(x+\mu^{1/\gamma})^{-\theta_n}}{x-(\alpha-\mu^{1/\gamma})}\,dx \tag{2.27}$$

We make use of the following standard integral [14]

$$\int_0^\infty\frac{(x+a)^{1-s}}{x-c}\,dx=\frac{\pi(a+c)^{1-s}}{\operatorname{tg}(s-1)\pi}-\frac{a^{2-s}}{a+c}\,B(s-2,1)\,F\left(2-s,\,1;\,3-s;\,\frac{a}{a+c}\right) \tag{2.28}$$

$$a>0,\quad c>0,\quad \operatorname{Re}\,s>0$$

where B is the beta function and F a hypergeometric function. Then from (2.27) we obtain

$$J_n=a^{-\theta_n}\left[\pi\operatorname{ctg}\theta_n\pi-(a/\mu^{1/\gamma})^{\theta_n-1}A_n\right],\qquad \alpha>\mu^{1/\gamma} \tag{2.29}$$

$$A_n=B(\theta_n-1,1)\,F(1,1-\theta_n;\;2-\theta_n;\;z),\qquad z=\alpha^{-1}\mu^{1/\gamma} \tag{2.30}$$

Inserting (2.29) and (2.30) into (2.26) we find, after some manipulations,

$$J(\alpha)=\varkappa\alpha^{-\nu}\sum_{n=0}^{\infty}\left[(-1)^n\frac{\Gamma(\theta_n)\cos\theta_n\pi}{\Gamma(\nu)\,n!}\,z^{\gamma n}-\frac{1}{\pi z}\,C_n z^n\right] \tag{2.31}$$

The coefficients C_n are obtained by expanding the second term of (2.29) into a power series in z. This is done with the help of a representation for a hypergeometric function [14]. As the result we have

$$A_n=\sum_{k=0}^{\infty}\frac{z^k}{\theta_n-k-1},\qquad C_n=\sum_{k=0}^{\infty}(-1)^n\frac{\Gamma(\theta_k)\sin\theta_k\pi}{\Gamma(\nu)\,k!\,(\theta-n-1)} \tag{2.32}$$

On the basis of relations (2.25), (2.29) and (1.19) we obtain the following representation for the resolvent:

$$R_\varkappa(t)=\frac{\mu^{1-\nu/\gamma}}{\pi}\int_0^\infty\left\{\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\theta_n)\sin\theta_n\pi}{\Gamma(\nu)\,n!}\,z^{n\gamma}\right\}\left\{1-\varkappa\frac{z^\nu}{\mu^{\nu/\gamma}}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\theta_n)\cos\theta_n\pi}{\Gamma(\nu)\,n!}\,z^{n\gamma}-\right.$$

$$\left.-\frac{1}{\pi}\,C_n z^{n-1}\right]^2+\varkappa^2\frac{z^{2\nu}}{\mu^{2\nu/\gamma}}\left[\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\theta_n)\sin\theta_n\pi}{\Gamma(\nu)\,n!}\,z^{n\gamma}\right]^2\Bigg\}^{-1}e^{-\frac{t\mu^{1/\gamma}}{z}}\,dz \tag{2.33}$$

The Neumann series for $R_\varkappa(t)$ follows from (2.33) by expanding the latter into a parametric power series in $\varkappa$. This boils down to expanding into a series a reciprocal of a quadratic trinomial. As this is quite tedious, we shall not go into it. We only note that by setting $\gamma=1$ in (2.31) we obtain, as we would expect, the resolvent (2.19). Indeed, in this case we firstly have

$$C_n=\sin\nu\pi\sum_{k=0}^{\infty}\frac{\nu(\nu+1)\ldots(\nu+k-1)}{(\nu+k-n-1)\,k!}\equiv 0 \tag{2.34}$$

and secondly

$$\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+\nu)\sin(n+\nu)\pi}{\Gamma(\nu)\,n!}\left(\frac{\mu}{\alpha}\right)^n=\sin\nu\pi\sum_{n=0}^{\infty}\frac{\nu(\nu+1)\ldots(\nu+n-1)}{n!}\left(\frac{\mu}{\alpha}\right)^n=$$

$$= \frac{\sin \nu\pi}{1-\mu/\alpha}$$

$$\sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(n+\nu)\cos(n+\nu)\pi}{\Gamma(\nu)\,n!}\left(\frac{\mu}{\alpha}\right)^n = \frac{\cos\nu\pi}{1-\mu/\alpha} \tag{2.35}$$

Inserting this into (2.33) we obtain (2.15) and after this, (2.19). Here we find that (2.34) represents a very nontrivial statement. To prove its validity we must use the following intermediate equation

$$\frac{1}{k-\nu} = \frac{1}{k} + \frac{\nu(\nu+1)}{k(k+1)} + \frac{\nu(\nu+1)(\nu+2)}{k(k+1)(k+2)} + \ldots \tag{2.36}$$

which can be extracted from the integral representation for a hypergeometric function [14].

The asymptotic properties of the approximate expressions for the resolvent etc. obtained earlier [15 - 17] are not considered here, nor is their accuracy analysed.

4. Boltzmann operator. Boltzmann, as well as a number of other investigators [3], have already noted that the deformation caused by a constant stress varies with time in a logarithmic manner. Recent precision measurements performed on a number of polymers [18, 19] confirm this point of view, particularly for the experiments of short duration. However the use of the logarithmic law in the computations was always hampered primarily by the lack of a suitably developed mathematical apparatus, and particularly by the lack of a resolvent. Attempts to obtain the latter by traditional methods have always failed just as in the case of the Bronskii-Slonimskii kernel [17].

Let us first obtain the properties of the spectrum of the logarithmic deformation law. Assuming in (1.3)

$$\eta(\lambda) = \eta_0 = \text{const}, \qquad 0 \leqslant \lambda < \infty \tag{2.37}$$

we find

$$R(t) = \eta_0/\varkappa t, \qquad \psi(t) = \eta_0 \ln t + C \tag{2.38}$$

i.e. a formal correspondence exists between the Boltzmann operator and a constant reduced delay spectrum. But it follows from (2.38) that neither the operator itself nor the logarithmic law, have any physical meaning when the whole time interval $[0, \infty)$ is taken into account, because of the singularity near the origin. This is also reflected in the fact that no relaxation spectrum exists for the chosen delay spectrum. This can easily be seen to follow from (1.20), as the improper integral appearing there diverges. There are two ways of overcoming this deficiency. The first one is based on altering the form of the Boltzmann operator. Let us assume [3]

$$R(t) = \frac{\eta_0}{\varkappa}\frac{1}{t+t_0}, \qquad t_0 > 0 \tag{2.39}$$

From (1.3) it follows that (2.38) is equivalent to the expression for the constant spectrum multiplied by a decaying exponential function

$$\eta(\lambda) = \eta_0 e^{-\lambda t_0} \tag{2.40}$$

Now

$$\int_0^{\infty} \frac{\eta(\lambda)}{\lambda-\alpha}\,d\lambda = -\eta(\alpha)\,\mathrm{Ei}(\alpha t_0) \tag{2.41}$$

where Ei (x) is an integral exponential function [14]. The representation for the resolvent is obtained from (1.3), (1.20), (2.39) and (2.40)

$$P(t) = q_0 \int_0^\infty \{[1 + q_0 e^{-\lambda t_0} \operatorname{Ei}(\lambda t_0)]^2 + \pi^2 q_0^2 e^{-2\lambda t_0}\}^{-1} e^{-\lambda(t+t_0)}\, d\lambda \tag{2.42}$$

From this it follows that the time enters the expression for $P(t)$ just as it did in (2.38), except that here it appears in the form of the sum $t + t_0$. The Neumann series for $P(t)$ is written in this case in the powers of the parameter q_0.

The second method is based on replacing the constant spectrum with a "block" spectrum [1,10]. Let

$$q(\lambda) = q_0 \quad (0 \leqslant \lambda \leqslant \lambda_0); \quad q(\lambda) = 0 \quad (\lambda > \lambda_0) \tag{2.43}$$

We choose $\lambda_0 = 1/t_0$. From (1.3) we find

$$R(t) = q_0 \varkappa^{-1} t^{-1} (1 - e^{-t/t_0}) \tag{2.44}$$

When $t_0 \to 0$ the kernel (2.44) tends the Boltzmann kernel (2.38). Omitting the intermediate calculations we give the resolvent for (2.44)

$$P(t) = q_0 t_0^{-1} \int_0^1 \{[1 - q_0 \ln(x^{-1} - 1)]^2 + \pi^2 q_0^2\}^{-1} \exp\frac{-xt}{t_0}\, dx \tag{2.45}$$

As before, the analysis of the resolvents (2.41) and (2.45) obtained is left aside.

Thus the proposed method of representing the resolvents opens new possibilities for a more flexible analytic description of the viscoelastic phenomena at both stages, this of studying the properties themselves and that of solving the boundary value problems.

BIBLIOGRAPHY

1. Gross B., Mathematical structure of the theories of viscoelasticity, Paris, Harmaun, 1953.
2. Blend D., Theory of Linear Viscoelasticity. M., "Mir", 1965.
3. Rabotnov Iu. N., Creep of the Construction Elements. M., "Nauka", 1966.
4. Rabotnov Iu. N., Equilibrium of an elastic medium with after-effect. PMM Vol.12, No.1, 1948.
5. Gromov V.G., On a method of describing the viscoelastic behavior of polymers. Dokl. Akad. Nauk SSSR, Vol.173, No.2, 1967.
6. Gromov V.G., On the problem of solving the boundary value problems of the linear viscoelasticity. Mechanics of Polymers., No.6, 1967.
7. Gromov V.G., Volterra operator algebra and its application to the viscoelastic problems., Dokl. Akad. Nauk SSSR, Vol.182, No.1, 1968.
8. Mikhlin S.G., Lectures on Linear Integral Equations., M., Fizmatgiz, 1959.
9. Ditkin V.A. and Prudnikov A.P., Operational Calculus., M., "Vysshaia shkola", 1965.
10. Ferry D., Viscoelastic Properties of Polymers., 2nd ed. Willey, 1970.
11. Shermergor T.D., Rheological characteristics of viscoelastic materials with asymmetric relaxation spectra., Inzh. zh. MTT, No.5, 1967.
12. Listovichnyi V.F. and Shermergor T.D., Creep of viscoelastic media with a kernel of the form of a degenerate hypergeometric function, Izv. Akad. Nauk SSSR, MTT, No.1, 1969.

13. Gnedenko B. V., Course of the Theory of Probability., ed. 2, M., Gostekhizdat, 1953.
14. Gradshtein I. S. and Ryzhik I. M., Tables of Integrals, Sums, Series and Products., M., Fizmatgiz, 1963.
15. Bronskii A. P., The occurence of the after-effect in a solid. PMM Vol. 5, No. 1, 1941.
16. Slonimskii G. L., On the law of deformation in the real materials. I. Zh. tekhn. fiz., Vol. 9, No. 20, 1939.
17. Koltunov M. A., Influence functions in the theory of shells with persistence properties. Coll. Studies in the Theory of Plates and Shells., Vol. 5, Kazan', Izd. Kazansk. Univ. 1967.
18. Gromov V. G., Kalinin V. A., Stepanenko Iu. P. and Shliafman E. M., Approximation of the experimental data obtained in investigating the viscoelastic properties of polymers., from the book: Abstracts from reports of the 3rd All-Union Symposium of the Theoretical and Applied Mechanics., M., "Nauka", 1968.
19. Stepanenko Iu. P., Problems encountered during the experimental investigation on the viscoelastic properties of polymers., Tr. Rostovsk. Univ. inzh. zh-d. transp., M., "Transport", No. 69, 1967.

Translated by L. K.

ON CERTAIN EXACT SOLUTIONS OF THE FOURIER EQUATION FOR REGIONS VARYING WITH TIME

PMM Vol. 35, No. 4, 1971, pp. 759-760
G. A. GRINBERG and V. A. KOSS
(Leningrad)
(Received January 11, 1971)

1. Let us use the method given in [1] to solve the first boundary value problem for the three-dimensional Fourier equation in Cartesian coordinates

$$\sum_{i=1}^{n} \frac{\partial^2 U}{\partial x_i^2} = \frac{\partial U}{\partial t} + f(x_1, \ldots, x_n, t) \qquad (n = 1, 2, 3) \tag{1.1}$$

defined on the domain bounded by coordinate planes moving in accordance with some rules $R_i^{(0)}(t)$ and $R_i^{(1)}(t)$ so that

$$x_i \in (R_i^{(0)}(t), R_i^{(1)}(t))$$

where i denote the coordinate axis number. Assume that the functions $R_i^{(0)}$ and $R_i^{(1)}$ possess continuous first and second order derivatives. We then obtain

$$\sum_{i=1}^{n} \frac{1}{\eta_i^2}\left[\frac{\partial^2 V}{\partial y_i^2} + \frac{1}{4}(\eta_i^2 \dot{\eta}_i^2 y_i^2 + 2\eta_i^2 R_i^{(0)\cdot\cdot} y_i)\, V\right] = \frac{\partial V}{\partial t} + \frac{f}{q} \tag{1.2}$$

$$U = q(y_1, \ldots, y_n, t)\, V(y_1, \ldots, y_n, t), \qquad y_i = \frac{x_i - R_i^{(0)}}{\eta_i} \qquad (\eta_i = R_i^{(1)} - R_i^{(0)}) \tag{1.3}$$